4

Wave Guides

4.1. Introduction

The transmission lines are capable of guiding energy from point to point in TEM mode, where as transmission of energy in TE and TM modes can be achieved by use of hollow conducting tubes known as *Wave guides*.

Wave guides may have rectangular or circular cross-section. In practice uniform cross-section is important for the same reason as transmission lines ; they are thus easier to manufacture and their properties are uniform and easier to calculate. In fact any configuration of electric and magnetic fields inside a wave guide must be solution of Maxwell's equations. These fields must also satisfy the boundary conditions imposed by the walls of the guide. It is of utmost importance to realise that conduction of energy takes place not through the walls of the guide; the guide walls only confine the energy. Actual energy conduction takes place through the dielectric filling the wave guide, usually air. In wave propagation it is necessary to speak in terms of electric and magnetic fields while discussing the properties of wave guide, unlike in transmission lines when we speak in terms of voltages and current. Moreover because the cross-sectional dimension of a wave guide must be of the same order as wavelength their use is economical at frequencies above 1 GHz.

Some advantages that the wave guides possess over coaxial lines are discussed here. Wave guides are simpler to manufacture than coaxial lines since the inner conductor is missing. Because

there is neither a inner conductor nor a dielectric in a wave guide flash over is less likely to occur. Therefore, the power handling capacity of wave guide is more, and may be about ten times as high as that of coaxial air dielectric rigid cables of similar dimensions. In a wave guide the propagation is by reflection from the walls, power losses in wave guides are lower than in comparable transmission lines.

Many different field configurations can be found to have no tangential component of electric field at the walls. Each such configuration if known as a *Mode*. The American system labels modes according to field component that behaves as it did in free space. Thus modes in which there is no component of electric field in the direction of propagation are called *Transverse Electric Mode or TE mode*; and modes with no component of magnetic field in direction of propagation are called *Transverse Magnetic Mode, or TM mode*.

The most striking difference between wave guides and transmission line is that a particular mode propagates down a wave guide with low attenuation only if the wavelength of the waves is less than some critical value determined by the dimensions and the geometry of the guide. If the wavelength corresponding to the operating frequency is greater than this critical *cut-off* value, the waves in the wave guides die out rapidly in amplitude even if walls of the guide have infinite conductivity. Different modes have different cut off wavelengths, the particular mode for which the cut off wavelength is the greatest is termed as *Dominant Mode*. In case of transmission lines, wavelength greater than cut off wavelength are passed.

4.2. Phase Velocity and Group Velocity in a Wave Guide

Phase velocity is velocity with which the electromagnetic wave changes phase. In free space this velocity is the same as the velocity with which it propagates, both being equal to the velocity of light. Group velocity is the velocity of the wave in a direction parallel to the conducting surface and is given by

$$v_g = v_c \sin \theta$$

where v_c is velocity of light.

This is clear from Fig. 4.1. Moreover phase velocity is equal to $v_c/\sin \theta$. With the help of these two equations we have

$$v_g v_p = v_c \sin \theta \cdot \frac{v_c}{\sin \theta} = v_c^2 \qquad \ldots(4.1)$$

Thus the product of the group and the phase velocity is the square of the velocity of light. In free space, however phase velocity and group velocity are equal. If λ_c is the cut off wavelength (the smallest free space wavelength that is just unable to propagate in the wave guide under given conditions) and λ is wavelength corresponding to operating frequency f then

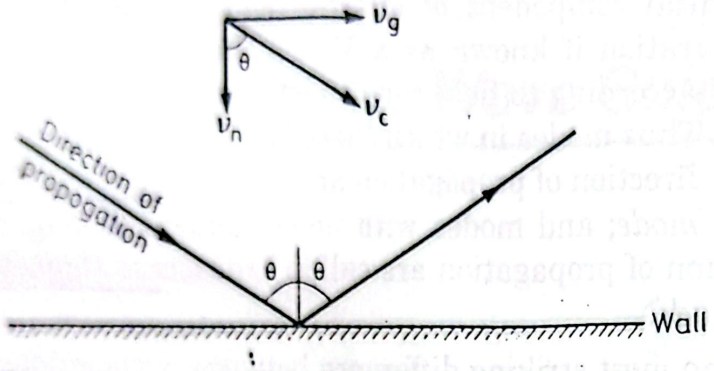

Fig. 4.1 Group velcoity in a wave guide (conducting surface)

$$v_p = f \lambda_p$$

[$\because \lambda_p$ = guide wave length]

$$= f \frac{\lambda}{\sqrt{\left(1 - \frac{\lambda}{\lambda_c}\right)^2}}$$

$$= \frac{v_c}{\sqrt{1 - \left(\frac{\lambda}{\lambda_c}\right)^2}} \quad \ldots(4.2$$

From Eqs. (4.1) and (4.2)

$$v_g = \frac{v_c^2}{v_p}$$

$$= v_c^2 \frac{1}{v_p}$$

$$= \frac{v_c^2 \sqrt{1 - \left(\frac{\lambda}{\lambda_c}\right)^2}}{v_c}$$

$$v_g = v_c \sqrt{1 - \left(\frac{\lambda}{\lambda_c}\right)^2} \quad \ldots(4.3)$$

Eq. (4.3) states that the group velocity in a wave guide is less than that in free space.

4.3. Mathematical Analysis of Rectangular Wave Guides

Rectangular wave guides have two dimensions a, b as shown in Fig. 4.2. To shart with we use Maxwell's equations to develop a wave equation in rectangular co-ordinates. Rectangular co-ordinate facilitate the application of boundary conditions to generalised equations. The restrictions are then imposed on wave travel in direction of guide. Then the choice is to made of the type of the mode to be analysed, it may be TE or TM mode. Then we end up with the set of equations giving the variations of each field component with respect to space and time. This method is very general and is used below in analysis of a hollow rectangular wave guide.

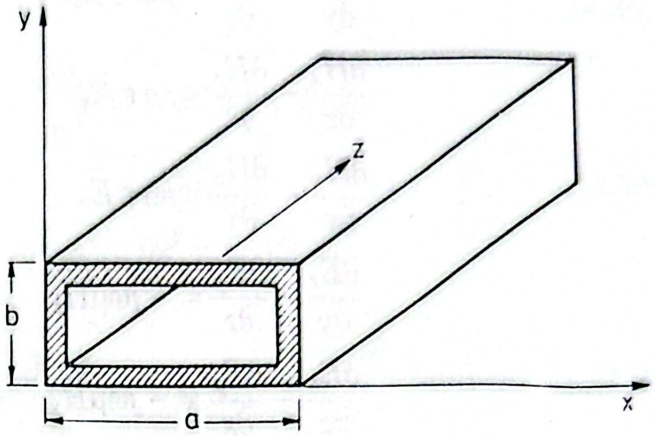

Fig. 4.2 Rectangular wave guide.

To determine the electromagnetic field configuration within the guide, Maxwell's equation are solved subject to appropriate boundary conditions at the walls of guide.

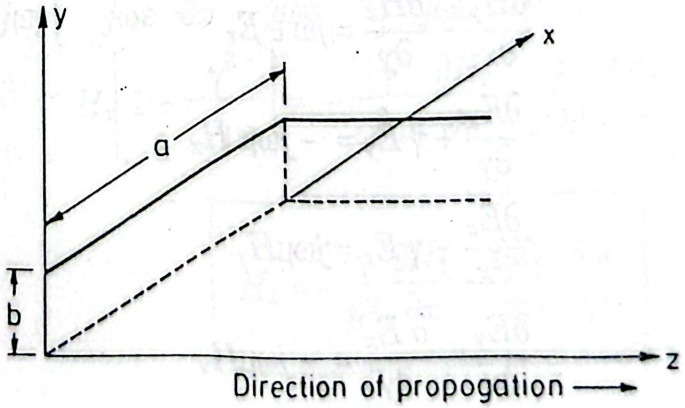

Fig. 4.3. Rectangular waveguide.

$$\nabla \times H = \varepsilon \frac{\partial E}{\partial t} = j\,\omega\,\varepsilon\,E \qquad ...(4.4)$$

$$\nabla \times E = -j\omega\mu H = -\mu \frac{\partial H}{\partial t} \qquad ...(4.5)$$

From Eq. (4.4) and (4.5)

$$\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial \zeta} = j\,\omega\,\varepsilon\,E_x \qquad \ldots(4.$$

$$\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = j\omega\,\varepsilon\,E_y \qquad \ldots(4.$$

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = j\omega\,\varepsilon\,E_z \qquad \ldots(4.$$

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -j\omega\mu H_x \qquad \ldots(4.$$

$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -j\omega\mu H_y \qquad \ldots(4.1$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -j\omega\mu H_z \qquad \ldots(4.1$$

We have the variation in z-direction according to exponenti law $e^{-\gamma z}$, so that the Maxwell's equations can be rewritten as

$$\frac{\partial H_z}{\partial y} + \gamma H_y = j\omega\,\varepsilon\,E_x \qquad \ldots(4.1$$

$$-\left(\frac{\partial H_z}{\partial x} + \gamma H_x\right) = j\omega\,\varepsilon\,E_y \qquad \ldots(4.1$$

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = j\omega\,\varepsilon\,E_z \qquad \ldots(4.1$$

$$\frac{\partial E_z}{\partial y} + \gamma E_y = -j\omega\mu H_x \qquad \ldots(4.1$$

$$\frac{\partial E_z}{\partial x} + \gamma E_x = j\omega\mu H_y \qquad \ldots(4.16$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -j\omega\mu H_y \qquad \ldots(4.17$$

Wave equations for E_z and H_z are given by

$$\nabla^2 E = \varepsilon\,\mu\,\frac{\partial^2 E}{\partial t^2} \qquad \ldots(4.1$$

or

$$\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + \frac{\partial^2 E_z}{\partial z^2} = -\omega^2\mu\,\varepsilon\,E_z \qquad \ldots(4.19$$

E_z is varying in accordance with both distance and time a $e^{-\gamma z}$ and $e^{j\omega t}$

Therefore,

$$\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + \gamma^2 E_z = -\omega^2 \mu \varepsilon E_z \qquad \ldots(4.20)$$

$$\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} = -(\omega^2 \mu \varepsilon + \gamma^2) E_z$$

$$= -h^2 E_z \qquad \ldots(4.21)$$

where $h^2 = (\omega^2 \mu\varepsilon + \gamma^2)$

Similarly,

$$\frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} = -h^2 H_z \qquad \ldots(4.22)$$

Now we try to express E_x, E_y, H_x, H_y in terms of E_x and H_z.

From Eq. (4.13)

$$H_x = -\frac{1}{\gamma} \frac{\partial H_z}{\partial x} - j \frac{\omega\varepsilon}{\gamma} E \qquad \ldots(4.23)$$

From Eq. (4.15)

$$H_x = -\frac{1}{j\mu\omega} \frac{\partial E_z}{\partial y} - \frac{\gamma}{j\omega\mu} E_y \qquad \ldots(4.24)$$

Substitute E_y from Eq. (4.13) into Eq. (4.24)

$$H_x = -\frac{\gamma}{j\omega\mu}\left[-\frac{1}{j\omega\varepsilon}\frac{\partial H_z}{\partial x} - \frac{\gamma}{j\omega\varepsilon} H_x\right] - \frac{1}{j\omega\mu}\frac{\partial E_z}{\partial y}$$

$$H_x\left[1 + \frac{\gamma^2}{\omega^2\mu\varepsilon}\right] = -\frac{\gamma}{\omega^2\mu\varepsilon}\frac{\partial H_z}{\partial x} - \frac{1}{j\omega\mu}\frac{\partial E_z}{\partial y}$$

$$\boxed{H_x = -\frac{\gamma}{h^2}\frac{\partial H_z}{\partial x} + \frac{\partial E_z}{\partial y}\frac{j\omega\varepsilon}{h^2}} \qquad \ldots(4.25)$$

From Eq. 4.16 and 4.12

$$H_y = \frac{1}{j\omega\mu}\frac{\partial E_z}{\partial x} + \frac{\gamma}{j\omega\mu} E_x \qquad \ldots(4.26)$$

$$E_x = \frac{1}{j\omega\varepsilon}\frac{\partial H_z}{\partial y} + \frac{\gamma}{j\omega\varepsilon} H_y$$

Substitute this value of E_x in Eq. (4.26)

$$\therefore \quad \frac{\gamma}{j\omega\mu}\left[\frac{1}{j\omega\varepsilon}\frac{\partial H_z}{\partial y} + \frac{\gamma}{j\omega\varepsilon} H_y\right] + \frac{1}{j\omega\mu}\frac{\partial E_z}{\partial x} = H_y$$

$$H_y = \frac{\omega^2\mu\varepsilon}{h^2}\left[-\frac{\gamma}{\omega^2\mu\varepsilon}\frac{\partial H_z}{\partial y} + \frac{1}{j\omega\mu}\frac{\partial E_z}{\partial x}\right]$$

$$H_y = -\frac{\gamma}{h^2}\frac{\partial H_z}{\partial y} - \frac{j\omega\varepsilon}{h^2}\frac{\partial E_z}{\partial x} \quad \ldots(4.27)$$

Similarly

$$E_x = \frac{1}{j\omega\varepsilon}\frac{\partial H_z}{\partial y} + \frac{\gamma}{j\omega\varepsilon}H_y$$

$$= \frac{1}{j\omega\varepsilon}\frac{\partial H_z}{\partial y} + \frac{\gamma}{j\omega\varepsilon}\left[-\frac{\gamma}{j\omega\mu}E_x + \frac{1}{j\omega\mu}\frac{\partial E_z}{\partial x}\right]$$

$$E_x\left[\frac{h^2}{\omega^2\mu\varepsilon}\right] = \frac{1}{j\omega\varepsilon}\frac{\partial H_z}{\partial y} - \frac{\gamma}{\omega^2\mu\varepsilon}\frac{\partial E_z}{\partial x}$$

$$\boxed{E_x = -\frac{j\omega\mu}{h^2}\frac{\partial H_z}{\partial y} - \frac{\gamma}{h^2}\frac{\partial E_z}{\partial x}} \quad \ldots(4.28)$$

$$E_y = -\frac{1}{j\omega\varepsilon}\left[\frac{\partial H_z}{\partial x} + \gamma H_x\right]$$

$$= -\frac{1}{j\omega\varepsilon}\frac{\partial H_z}{\partial x} + j\frac{\gamma}{\omega\varepsilon}\left[\frac{j\omega\varepsilon}{\gamma^2}\frac{\partial E_z}{\partial y} - \frac{\gamma}{h^2}\frac{\partial H_z}{\partial x}\right]$$

$$E_y = -\frac{\gamma}{h^2}\frac{\partial E_z}{\partial y} + \frac{j\omega\mu}{h^2}\frac{\partial H_z}{\partial x} \quad \ldots(4.29)$$

Now three cases arise.

(1) When $E_z = H_z = 0$.

This corresponds to a TEM wave, which cannot exist inside the wave guide.

(A detailed explanation follows)

(2) When $E_z = 0, H_z \neq 0$

TE wave exists in a wave guide.

(3) When $H_z = 0, E_z \neq 0$

TM wave exists in the wave guide.

We now proceed to analyse both the TE and TM waves separately.

.4. Transmission of TM Waves in Rectangular Wave Guide

For case $H_z = 0, E_z \neq 0$

From Eqs. (4.25), (4.27), (4.28) and (4.29)

$$H_x = \frac{j\omega\varepsilon}{h^2}\frac{\partial E_z}{\partial y} \quad ...(4.30)$$

$$H_y = -\frac{j\omega\varepsilon}{h^2}\frac{\partial E_z}{\partial x} \quad ...(4.31)$$

$$E_x = -\frac{\gamma}{h^2}\frac{\partial E_z}{\partial x} \quad ...(4.32)$$

$$E_y = \frac{-\gamma}{h^2}\frac{\partial E_z}{\partial y} \quad ...(4.33)$$

Wave equation from Eq. (4.21)

$$\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} = -h^2 E_z$$

Since the surface is a perfect conductor, hence the tangential component of electric field is zero.

Boundary conditions are :

At $y = 0$

and $y = b,$

$E_x = E_y = 0$

At $x = 0$

and $x = a$

$E_y = E_z = 0.$

The solution of wave equation [Eq. (4.21)] is

$E_z = XY$

where X = Function of x only

Y = Function of y only.

So the Eq. (4.21) can be rewritten as

$$\frac{d^2 E_z}{dx^2} = Y\frac{d^2 X}{dx^2}$$

$$\frac{\partial^2 E_z}{\partial y^2} = X\frac{\partial^2 Y}{\partial y^2}$$

So, $$Y\frac{\partial^2 X}{\partial x^2} + X\frac{\partial^2 Y}{\partial y^2} = -h^2 XY$$

or $$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} + h^2 = 0$$

[Dividing throughout by XY]

$$\text{or} \quad \frac{1}{X}\frac{\partial^2 X}{\partial x^2} + h^2 = -\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} \qquad \ldots(4.34)$$

$$\text{let} \quad \frac{1}{X}\frac{\partial^2 X}{\partial x^2} + h^2 = A^2$$

$$\Rightarrow \quad \frac{1}{X}\frac{\partial^2 X}{\partial x^2} = A^2 - h^2 = -B^2 \qquad \ldots(4.35)$$

where $$B^2 = h^2 - A^2$$

$$A^2 = -\frac{1}{Y}\frac{\partial^2 Y}{dy}. \qquad \ldots(4.36)$$

The solutions of standard differential equations of second order of Eqs. (4.35), (4.36) are given as

$$X = C_1 \cos Bx + C_2 \sin Bx$$

$$\dot{Y} = C_3 \cos Ay + C_4 \sin Ay.$$

where C_1, C_2, C_3, and C_4 are constants to be determined

$$E_z = XY$$

$$= (C_1 \cos Bx + C_2 \sin Bx)$$

$$(C_3 \cos Ay + C_4 \sin Ay)$$

$$= (C_1C_3 \cos Bx \cos Ay + C_2C_3 \sin Bx$$

$$\cos Ay + C_1C_4 \cos Bx \sin Ay + C_2C_4 \sin Bx \sin Ay)$$

Now applying the boundary conditions

$x = 0, \Rightarrow E_z = 0 = C_1C_3 \cos Ay + C_1C_4 \sin Ay$

$C_1 = 0$

$$\Rightarrow E_z = C_2 (C_3 \sin Bx \cos Ay + C_4 \sin Bx \sin Ay) \qquad \ldots(4.37)$$

At $$y = 0$$

$$E_z = 0 = C_2C_3 \sin Bx$$

$C_3 = 0$, because we see from Eq. (4.37) that when $C_2 = 0$ then both the terms become zero.

So $$E_z = C_2C_4 \sin Bx \sin Ay$$

$$E_z = C \sin Bx \sin Ay \qquad \ldots(4.38)$$

at $$x = a, E_z = 0$$

$$E_z = 0 = C \sin Ba \sin Ay$$

$$\Rightarrow \quad Ba = m\pi$$

$$B = \frac{m\pi}{a}, \text{ where } m \text{ is any integer}$$

at $$y = b$$

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} + h^2 = -\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} \qquad \text{...(4.3}$$

or

let $$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} + h^2 = A^2$$

$$\Rightarrow \quad \frac{1}{X}\frac{\partial^2 X}{\partial x^2} = A^2 - h^2 = -B^2 \qquad \text{...(4.3}$$

where $$B^2 = h^2 - A^2$$

$$\therefore \quad A^2 = -\frac{1}{Y}\frac{\partial^2 Y}{dy}. \qquad \text{...(4.3}$$

The solutions of standard differential equations of second orde of Eqs. (4.35), (4.36) are given as

$$X = C_1 \cos Bx + C_2 \sin Bx$$

$$Y = C_3 \cos Ay + C_4 \sin Ay.$$

where C_1, C_2, C_3, and C_4 are constants to be determined

$$E_z = XY$$

$$= (C_1 \cos Bx + C_2 \sin Bx)$$

$$(C_3 \cos Ay + C_4 \sin Ay)$$

$$= (C_1C_3 \cos Bx \cos Ay + C_2C_3 \sin Bx$$

$$\cos Ay + C_1C_4 \cos Bx \sin Ay + C_2C_4 \sin Bx \sin Ay$$

Now applying the boundary conditions

$x = 0, \Rightarrow E_z = 0 = C_1C_3 \cos Ay + C_1C_4 \sin Ay$

$C_1 = 0$

$\Rightarrow E_z = C_2 (C_3 \sin Bx \cos Ay + C_4 \sin Bx \sin Ay)$ ...(4.37

At $y = 0$

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So $$E_z = C_2C_4 \sin Bx \sin Ay$$

$$E_z = C \sin Bx \sin Ay \qquad \text{...(4.38)}$$

at $x = a, E_z = 0$

$$E_z = 0 = C \sin Ba \sin Ay$$

$$\Rightarrow \quad Ba = m\pi$$

$$B = \frac{m\pi}{a}, \text{ where } m \text{ is any integer}$$

at $y = b$

$$E_z = 0 = C \sin Bx \sin Ab$$

$$\Rightarrow \quad Ab = n\pi$$

or $$A = \frac{n\pi}{b}, \text{ where } n \text{ is any integer.}$$

$$E_z = \left[C \sin \frac{m\pi}{a} x \sin \frac{n\pi}{b} y \right].$$

Substituting the value of E_z in Eqs. (4.30), (4.31), (4.32) and (4.33).

$$H_x = \frac{j\omega\epsilon}{h^2} C \frac{n\pi}{b} \sin \frac{m\pi}{a} x \cos \frac{n\pi}{b} y \qquad ...(4.39)$$

$$H_y = \frac{-j\omega\epsilon}{h^2} C \frac{m\pi}{a} \cos \frac{m\pi}{a} x \sin \frac{n\pi}{b} y \qquad ...(4.40)$$

$$E_x = -\frac{\gamma}{h^2} C \frac{m\pi}{a} \cos \frac{m\pi}{a} x \sin \frac{n\pi}{b} y \qquad ...(4.41)$$

$$E_y = -\frac{\gamma}{h^2} C \frac{n\pi}{b} \cos \frac{n\pi}{b} y \sin \frac{m\pi}{a} x \qquad ...(4.42)$$

Eqs. (4.39), (4.40), (4.41) and (4.42) have to be multiplied by $e^{-(j\omega t - \gamma z)}$

where $\gamma = \alpha + j\beta$, $\alpha = 0$ for perfect propagation.

4.4.1. Propagation properties and cut off Frequencies for Rectangular wave guide

$$A^2 + B^2 = h^2$$

$$\gamma^2 + \omega^2\mu\epsilon = h^2$$

$$\gamma = \sqrt{h^2 - \omega^2\mu\epsilon}$$

$$= \sqrt{A^2 + B^2 - \omega^2\mu\epsilon}$$

$$\gamma = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - \omega^2\mu\epsilon}$$

When γ is real, then α has a finite value.

If $\beta = 0$, wave cannot propagate through a rectangular wave guide.

When γ is imaginary, then the wave propagates through wave guide without attenuation, *i.e.* $\alpha = 0$. Hence for propagation through rectangular wave guide.

$$\boxed{\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right] < \omega^2\mu\epsilon}$$

In the above inequality, only ω can be adjusted on R.H.S.

There will be some values of ω for which the wave will be attenuated inside the wave guide and for some values of ω, it will propagate through the wave guide

$$\omega_e \mp \sqrt{\frac{1}{\mu\varepsilon}\left\{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right\}}$$

$f_c = \frac{1}{2\pi}\omega_c$ = critical frequency or cutoff frequency

for $\omega < \omega_c$, the wave won't propagate

$\omega > \omega_c$, wave will propagate through wave guide as γ is imaginary.

So that $$\beta = \sqrt{\omega^2\mu\varepsilon - \left\{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right\}}$$

Phase velocity

$$= \frac{\omega}{\beta} = \frac{\omega}{\left[\omega^2\mu\varepsilon - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2\right]^{1/2}}$$

Substituting $\gamma = jB$ in Eqs. (4.39) to (4.42), we get

$$E_x = -\frac{j\beta}{h^2}\frac{m\pi}{a}\cos\frac{m\pi}{a}x\sin\frac{n\pi}{b}y \quad ...(4.43)$$

$$E_y = -\frac{j\beta}{h^2}C\frac{n\pi}{b}\sin\frac{m\pi}{a}x\cos\frac{n\pi}{b}y \quad ...(4.44)$$

$$H_x = \frac{j\omega C\varepsilon}{h^2}\cdot\frac{n\pi}{b}\sin\frac{m\pi}{a}x\cos\frac{n\pi}{b}y \quad ...(4.45)$$

$$H_y = \frac{j\omega C\varepsilon}{h^2}\cdot\frac{m\pi}{a}\cos\frac{m\pi}{a}x\sin\frac{n\pi}{b}y \quad ...(4.46)$$

Any wave guide behaves like a high pass fitter, for $f > f_c$, there is no attenuation, $f < f_c$, no part of the wave is propagated.

For the case under consideration (perfectly conducting walls), the attenuation constant $\propto = 0$ for all frequencies such that $f > f_c$. For these frequencies

$$\beta = \sqrt{\omega^2\mu\varepsilon - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2} \quad ...(4.47)$$

The value of ω_c is given by

$$\omega_c = \frac{1}{\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} \quad ...(4.48)$$

The cut off frequency, *i.e.* the frequency below which wave propagation will not occur, is

$$f_c = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} \qquad ...(4.49)$$

The corresponding cut-off wave length is

$$\lambda_c = \frac{2}{\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}} \qquad ...(4.50)$$

So $$f_c \cdot \lambda_c = v_c.$$

The velocity of propagation will be given by

$$\bar{v} = \frac{\omega}{\beta} = \frac{\omega}{\sqrt{\omega^2\mu\varepsilon - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}} \qquad ...(4.51)$$

The last expression indicates that the velocity of propagation of the wave in the guide is greater than the phase velocity in free space. As frequency is increased above cut-off, the phase velocity decreases from an infinitely large value and approaches v_c, (the velocity in free space), as the frequency increases without limit.

Since the wave length in the guide is given by $\bar{\lambda} = \bar{v}/f$, it will be longer than the corresponding free space wavelength.

From Eqs. (4.51), we have

$$\bar{\lambda} = \frac{2\pi}{\sqrt{\omega^2\mu\varepsilon - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}} \qquad ...(4.52)$$

In the above expressions the only restriction on m and n is that they should be integers. However by Eqs. (4.43) to (4.46) it is seen that if either m or n is zero, the fields will be identically zero.

Therefore, the lowest possible value for either m or n (for TM waves) is unity. From Eq. (4.49) it is evident that the lowest cut-off frequency will occur for $m = n = 1$. Substituting these values in Eqs. (4.43) to (4.46) gives the field for the lowest frequency TM wave which can be propagated through the guide. These waves (larger values of m and n) require higher frequencies in order to be propagated along a guide of given dimensions.

4.5. Transelectric Waves (TE Waves)

We proceed to derive the various relations, for TE waves in the foregoing discussion.

For *TE* waves, $H_z \neq 0$ and $E_z = 0$.

Following exactly similar steps as for TM waves we can find in this case too, the various equations as follows :

$$\left.\begin{aligned} \frac{\partial H_z}{\partial y} - \gamma H_y &= j\omega\varepsilon E_x \\ -\gamma H_x - \frac{\partial H_z}{\partial x} &= j\omega\varepsilon E_y \\ \gamma E_y &= -j\omega\mu H_x \\ -\gamma E_x &= -j\omega\mu H_y \\ \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} &= -j\omega\mu H_z \end{aligned}\right\} \quad \ldots(4.5$$

and the wave equation is

$$\frac{\partial^2 H_z}{\partial x^2} = -h^2 H_z.$$

where the symbols have their usual meaning as explained before

The general solution of the wave equation given above, before, is

$$\left.\begin{aligned} H_z &= C \cos Bx \cos Ay \\ H_x &= \frac{C\gamma}{h^2} B \sin Bx \cos Ay \\ H_y &= \frac{C\gamma}{h^2} A \cos Bx \sin Ay \\ E_x &= \frac{j\omega\mu}{h^2} CA \cos Bx \sin Ay \\ E_y &= -\frac{j\omega\mu C}{h^2} B \sin Bx \cos Ay \end{aligned}\right\} \quad \ldots(4.5$$

In above equations

$$A = \frac{n\pi}{b}$$

and

$$B = \frac{m\pi}{a}$$

Substituting the values of A and B in the Eq. (4.54), we get

$$\left.\begin{aligned} H_z &= C\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b} \\ H_x &= \frac{C\gamma}{h^2}\cdot\frac{m\pi}{a}\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b} \\ H_y &= \frac{C\gamma}{h^2}\frac{n\pi}{b}\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \\ E_x &= \frac{C\,j\omega\varepsilon}{h^2}\frac{n\pi}{b}\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \\ E_y &= -C\frac{j\omega\varepsilon}{h^2}\frac{m\pi}{a}\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b} \end{aligned}\right\} \quad \ldots(4.55)$$

$$h^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 = (\gamma^2 + \omega^2\mu\varepsilon)$$

m and n may have any integral value.

The equations for various constants $\bar{\beta}$, f_c, λ_c, $\bar{v}$ and λ are same for TE as well as TM waves. We may rewrite them for simplicity sake as

$$\left.\begin{aligned} \beta &= \sqrt{\omega^2\mu\varepsilon - \left\{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right\}} \\ f_c &= \frac{1}{\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} \\ \lambda_c &= \frac{2\pi}{\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}} \\ \bar{v} &= \frac{\omega}{\sqrt{\omega^2\mu\varepsilon - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}} \\ \lambda &= \frac{2\pi}{\sqrt{\omega^2\mu\varepsilon - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}} \end{aligned}\right\} \quad \ldots(4.56)$$

where symbols have their usual meaning.

From Eq. (4.53) of TE waves it is found possible to make either m or n equal to zero without causing all the fields to vanish, *i.e.* a lower order is possible then in TM wave case. The lowest order TE

wave in rectangular wave guides is therefore, TE_{10} wave. This wa which has the lowest cut off frequency is called the dominant mod

It is seen that the subscripts m and n represent the number half period variation of the fields along X and Y co-ordinat respectively. For practical reasons, in most experimental work wi rectangular wave guides, the dominant TE_{10} wave is used. substituting $m = 1$, $n = 0$ in general equations for f_c, and λ_c, we g

$$\lambda_c = 2a; \qquad \ldots(4.57$$

$$f_c = \frac{v_c}{2a} \qquad \ldots(4.57$$

$$\beta = \sqrt{\omega^2 \mu\varepsilon - \left(\frac{\pi}{a}\right)^2} \qquad \ldots(4.57$$

For the TE_{10} wave, the cut-off frequency is that frequency which the corresponding (free-space) half wavelength is equal to t width of the guide.

As can be seen from above, for TE_{10} mode wave the cut frequency is independent of the dimension b of the wave guide.

Fig. 4.4 indicates the various field configurations for lower ord TE and TM waves in rectangular wave guides.

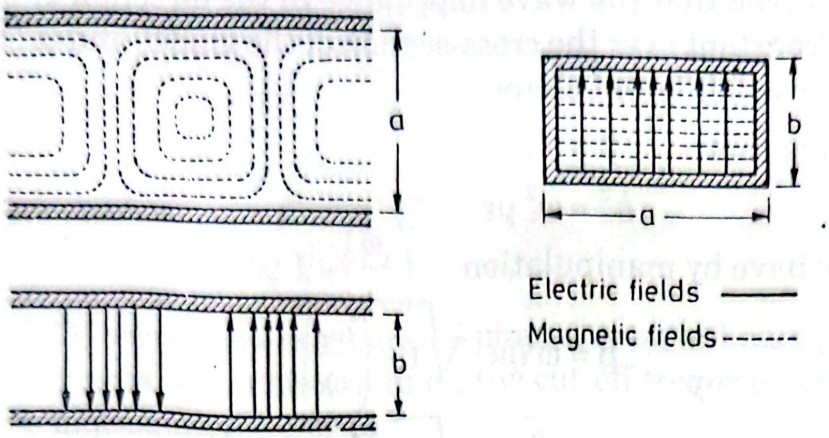
It is possible for several modes to exist simultaneously in guide if the frequency is above cut off for those particular mod However, the guide dimensions are often chosen so that only t dominant mode can exist.

Wave impedances and characteristic impedances. F waves guided by transmission lines or wave guides, interesti thing is the wave impedance which is seen when looking in t direction of propagation, that is, along say z axis.

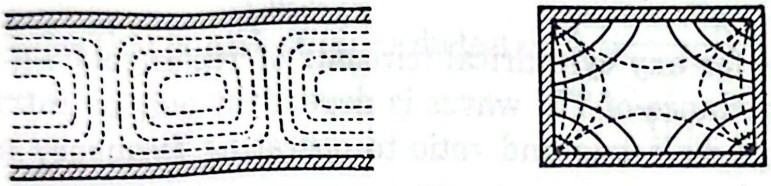
We know from equations of the transverse field components a TM wave in a rectangular guide that

$$Z_{z(TM)} = \frac{\overline{E}}{\overline{H}} = \frac{E_x}{H_y} = -\frac{E_y}{H_x} = \frac{\sqrt{E_x^2 + H_y^2}}{\sqrt{H_y^2 + H_x^2}}$$

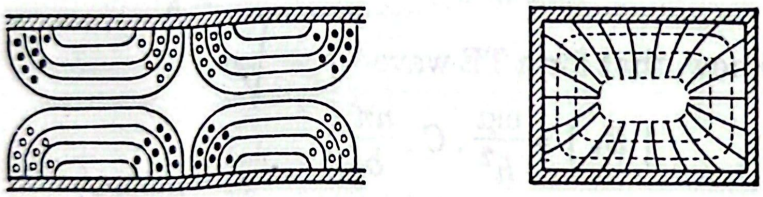
That is, the wave impedance tooking in z direction is the ra of transverse electric field strength to the total magnetic fi strength.



TE_{10} WAVE



TE_{11} wave



TM_{11} WAVE

Electric fields : Solid
Magnetic fields : Dashed

Fig. 4.4. Lower order modes in rectangular wave guides.

We know,

$$\left.\begin{aligned} |E_x| &= \frac{\beta}{h^2} \cdot C \frac{m\pi}{a} \\ |E_y| &= \frac{\beta}{h^2} \cdot C \frac{n\pi}{b} \\ |H_x| &= \frac{\omega\varepsilon}{h^2} \cdot C \frac{n\pi}{b} \\ |H_y| &= \frac{\omega\varepsilon}{h^2} \cdot C \frac{m\pi}{a} \end{aligned}\right] \Rightarrow \frac{E_x}{H_y} = \frac{\beta}{\omega\varepsilon} = \frac{E_y}{H_x} = Z_{z(TM)}$$

It is seen that the wave impedance in the direction of propagation is 'constant over the cross-section of the guide', and is the same for guides of different shape.

Recalling that, $\beta = \sqrt{\omega^2 \mu\varepsilon - h^2}$

and $h^2 = \omega_c^2 \mu\varepsilon$

We have by manipulation

$$\beta = \omega\sqrt{\mu\varepsilon}\sqrt{1 - \left(\frac{\omega_c}{\omega}\right)^2} \qquad \ldots(4.58$$

$$\Rightarrow \quad Z_{z(TM)} = \frac{\beta}{\omega\varepsilon} = \sqrt{\frac{\mu}{\varepsilon}}\sqrt{1 - \left(\frac{\omega_c}{\omega}\right)^2}$$

$$= \eta_{\text{dielectric}}\sqrt{1 - \left(\frac{\omega_c}{\omega}\right)^2} \qquad \ldots(4.59$$

Thus, for any cylindrical (circular or rectangular) guide, the wave impedance of TM waves is dependent only on intrinsic impedance of dielectric and ratio to operating frequency of cut off frequency.

For TE waves the same conclusion can be reached as shown below.

We know, that for a TE wave

$$|E_x| = \frac{\omega\mu}{h^2} \cdot C \cdot \frac{n\pi}{b};$$

$$|H_y| = \frac{\beta}{h^2} \cdot C \cdot \frac{n\pi}{b}$$

$$|E_y| = \frac{\omega\mu}{h^2} \cdot C \cdot \frac{m\pi}{a};$$

$$|H_x| = \frac{\beta}{h^2} \cdot C \cdot \frac{m\pi}{a}.$$

$$= Z_{Z(TE)} = \frac{E_y}{H_x}$$

$$= \frac{E_x}{H_y} = \frac{\omega\mu}{\beta}$$

$$= \frac{\omega\mu}{\omega\sqrt{\mu\varepsilon}\sqrt{1 - \left(\frac{\omega_c}{\omega}\right)^2}}$$

$$= \frac{\sqrt{\mu/\varepsilon}}{\sqrt{1-\left(\frac{\omega_c}{\omega}\right)^2}}$$

$$= \frac{\eta_{dielectric}}{\sqrt{1-\left(\frac{\omega_c}{\omega}\right)^2}} \qquad ...(4.60)$$

For TEM waves between parallel planes, or on ordinary parallel wire or coaxial transmission lines, the cut-off frequency is zero, and wave impedance reduces to

$$Z_{Z(TEM)} = \eta \qquad ...(4.61)$$

It is important to note the behaviour of the wave guide when a dielectric (other than air) is introduced inside the guide.

We know that in case of air as dielectric, $\bar{\lambda} = \dfrac{\lambda}{\sqrt{1-\left(\frac{\lambda}{\lambda_c}\right)^2}}$

and in case of a dielectric with constant ε_r

$$\bar{\lambda} = \frac{\lambda}{\sqrt{\varepsilon_r - \left(\frac{\lambda}{\lambda_c}\right)^2}}, \text{ and that} \qquad ...(4.62)$$

$$\bar{\lambda}_{dielectric} = \frac{\bar{\lambda}_{air}}{\sqrt{\varepsilon_r}}.$$

Since $\varepsilon_r \geq 1$, $\lambda_d \leq \lambda_{air}$ and hence frequencies less than cut-off value f_c can pass through the same guide.

Problem 4.1. *Show that for a TE_{10} mode, a frequency of 8 GHz will pass through a wave-guide of dimensions $a = 1.5$ cm, $b = 1$ cm if a dielectric with $\varepsilon_r = 4$ is inserted inside the guide.*

Solution. Let the wave guide cross-section be as shown alongside. Since $a = 1.5$ cm, cut-off wave-length of TE_{10} mode $\lambda_c = 2a = 3$ cm.

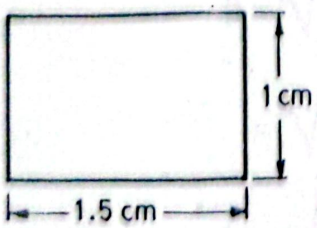


Fig 4.5

Hence $f_c = \dfrac{3 \times 10^{10}}{3} = 10$ GHz.

As is evident, a frequency of 8 GHz which is less than 10 GHz (cut-off frequency) will not ordinarily pass through the guide.

But when a dielectric is inserted inside the guide, this frequency of 8 GHz will readily pass through the same guide. λ for 8 GHz $= \dfrac{3 \times 10^{10}}{8 \times 10^{9}} = 3.75$ cm in air.

We know that

$$\lambda_{dielectric} = \frac{\lambda_{air}}{\sqrt{\varepsilon}}$$

$$= \frac{3.75}{\sqrt{4}}$$

$$= 1.87 \text{ cm.}$$

which is less than 3 cm and hence 8 GHz frequency will pass through the same guide.

Problem 4.2. *A rectangular wave guide is to be designed to propagate 10 GHz signal under the dominant mode.*

(a) Estimate the dimensions of wave guide

(b) Calculate $\bar{v}$, guide wave length, γ.

Solution. $f = 10$ GHz

$$\lambda = \frac{3 \times 10^{10}}{10^{10}} = 3 \text{ cm}$$

$$\lambda > a > \lambda/2$$

$$3 > a > 1.5$$

$\therefore$ let $a = 2$ cm

$$b < \lambda/2$$

$$b = 1 \text{ cm}$$

$$(\lambda_c)_{TE_{10}} = 4 \text{ cm } (2a = 2 \times 2 = 4 \text{ cm})$$

$$\bar{\lambda} = \frac{\lambda}{\sqrt{1 - (\lambda/\lambda_c)^2}}$$

$$= \frac{3 \times 4}{\sqrt{7}} = 4.5 \text{ cm.}$$

$$\bar{v} = \frac{v_0}{\sqrt{1 - (\lambda/\lambda_c)^2}}$$

$$= \frac{3 \times 10^{10} \times 4}{\sqrt{7}}$$

$$= 4.5 \times 10^{10} \text{ cm/sec.}$$

$$\beta = \omega \sqrt{\mu\varepsilon} \sqrt{1 - \left(\frac{\omega_c}{\omega}\right)^2}$$

$[\gamma = \alpha + j\beta$ when $\omega > \omega_c, \gamma = \beta]$

$$\frac{\omega_c}{\omega} = \frac{f_c}{f} = \frac{\lambda}{\lambda_c} = \frac{3}{4}$$

$$\beta = 2\pi \times 10^{10} \sqrt{\frac{10^{-9}}{36\pi} \times 4\pi \times 10^{-7}} \sqrt{1 - \left(\frac{3}{4}\right)^2}.$$

$$= 2\pi \times 10^{10} \times \frac{10^{-8}}{3} \times \frac{\sqrt{7}}{4}.$$

$$= 138.5 \text{ rad/m}$$

$$= 1.385 \text{ rad/cm. } \textbf{Ans.}$$

Problem 4.3. *A hollow rectangular guide has dimensions $a = 4$ cm , $b = 2.6$ cm. The frequency of the impressed signal is 7 GHz. Compute the following for possible modes (a) λ_c (b) $\overline{\lambda}$, β, $\overline{v}$; (c) Z_{TE} , Z_{TM}.*

Solution. Wave length of impressed signal is

$$= \frac{3 \times 10^8}{7 \times 10^9} = 4.3 \text{ cm.}$$

(*i*) TE_{10} critical wave length is

(*a*) $$\lambda_c = \frac{2}{\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}}$$

For $$TE_{10} = \lambda_c = 2a$$

$$= 2 \times 4 = 8 \text{ cm}$$

> 4.3 cm (Possible mode). **Ans.**

(*b*) $$\beta = \sqrt{\omega^2\mu\varepsilon - \left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]}$$

$$= \sqrt{\frac{(2\pi \times 7 \times 10^9)^2 \times 4\pi \times 10^{-7}}{36\pi \times 10^9} - \left[\frac{\pi}{0.04}\right]^2}$$

$$= \sqrt{4\pi^2 \times 49 \times 10^{18} \times \frac{10^{-16}}{9} - \frac{\pi^2}{0.0016}}$$

$$= \pi \times 39.4$$

$$= 124 \text{ rad/m}$$

$= 1.24$ rad/cm. **Ans.**

$$\bar{v} = \frac{\omega}{\beta} = \frac{2\pi \times 7 \times 100^9}{\pi \times 39.4}$$

$= 3.55 \times 10^8$ m/sec. **Ans.**

$$\lambda = \frac{v}{f} = \frac{2\pi \times 100}{\pi \times 39.4}$$

$$= \frac{200}{39.4} = 5.07 \text{ cm. Ans.}$$

(*ii*) Mode TE_{01}

(*a*) $\lambda_c = 2b = 2 \times 2.6$

$= 5.2$ cm > 4.3 cm (possible mode)

(*b*) $$\beta = \pi \sqrt{2180 - \frac{1}{(0.026)^2}}$$

$= \pi \sqrt{2180 - 1480}$

$= \pi\sqrt{700} = 83.1$ rad/m

$= 26.45\ \pi$ rad/m

$= 0.831$ rad/cm. **Ans.**

$$\bar{v} = \frac{\omega}{\beta} = \frac{2\pi \times 7 \times 10^9}{26.45\pi}$$

$= 5.3 \times 10^8$ m/sec. **Ans.**

$$\lambda = \frac{\bar{v}}{f} = \frac{2\pi \times 100}{26.45\pi}$$

$= 7.56$ cm. **Ans.**

(*iii*) $TE_{11} = TM_{11}$

(*a*) $$\lambda_c = \frac{2ab}{\sqrt{a^2 + b^2}} = \frac{20.8}{\sqrt{6.76 + 16}}$$

$= 4.36$ cm > 4.3 (possible modes)

(*b*) $$\beta = \sqrt{2180\pi^2 - \left[\left(\frac{\pi}{0.04}\right)^2 + \left(\frac{\pi}{0.026}\right)^2\right]}$$

$= \pi \sqrt{2180 - [625 + 1480]}$

$= \pi\sqrt{75} = 0.272$ rad/cm. **Ans.**

$$\bar{v} = \frac{\omega}{\beta} = \frac{2\pi \times 7 \times 10^9}{8.66\pi}$$

$= 1.615 \times 10^9$ m/sec. **Ans.**

$$\lambda = \frac{\bar{v}}{f} = \frac{2\pi \times 100}{8.66\pi}$$

$$= 23.1 \text{ cm. Ans.}$$

For TE_{28}, $\lambda_c = a = 4$ cm < 4.3 not possible

TE_{02}, $\lambda_c = b = 2.6 < 4.3$ not possible.

$$TE_{12} = TM_{12}, \lambda_c = \frac{20.8}{\sqrt{6.76 + 64}}$$

< 4.3 not possible.

$$TM_{21} = TE_{21}, \lambda_c = \frac{20.8}{\sqrt{27.04 + 16}}$$

< 4.3 not possible

(c) for TE_{10}

$$Z_{TM} = \sqrt{\frac{\mu}{\varepsilon}} \sqrt{1 - \left(\frac{\omega_c}{\omega}\right)^2}$$

$$\frac{\omega_c}{\omega} = \frac{2\pi \times 3.75 \times 10^9}{2\pi \times 7 \times 10^9}$$

$$= 0.535$$

$$Z_{TM} = 377 \sqrt{1 - 0.286}$$

$$= 318.5 \ \Omega. \text{ Ans.}$$

$$Z_{TR} = \frac{\sqrt{\mu/\varepsilon}}{\sqrt{1 - \left(\frac{\omega_c}{\omega}\right)^2}}$$

$$= \frac{377}{\sqrt{0.714}}$$

$$= 446 \ \Omega. \text{ Ans.}$$

For $TE_{01} = f_c = \dfrac{3 \times 10^{10}}{5.2}$

$$= 5.22 \text{ GHz.}$$

$$\frac{\omega_c}{\omega} = \frac{5.77}{7}$$

$$= 0.825.$$

$$Z_{TM} = 377\sqrt{1 - (0.825)^2}$$

$$= 377\sqrt{1 - 0.68}$$

$$= 214 \ \Omega. \text{ Ans.}$$

$$Z_{TM} = \frac{377}{\sqrt{0.32}} = 666\ \Omega.\ \textbf{Ans.}$$

For $TE_{11} = TM_{11}$

$$f_c = \frac{3 \times 10^{10}}{4.36}$$

$$= 6.88\ \text{GHz}$$

$$\frac{\omega}{\omega_c} = \frac{6.88}{7}$$

$$= 0.983$$

$$Z_{TM} = 377\sqrt{1 - (0.983)^2}$$

$$= 377\sqrt{1 - 0.963}$$

$$= 67.5\ \Omega.\ \textbf{Ans.}$$

$$Z_{TM} = \frac{377}{\sqrt{1 - 0.963}}$$

$$= 2100\ \Omega.\ \textbf{Ans.}$$

Problem 4.4. *A hollow rectangular wave guide has dimensions $a = 4$ cm , $b = 2$ cm, calculate the amount of attenuation if the frequency is 3 GHz.*

Solution. The critical frequency of the given waveguide is 3.75 GHz.

$\therefore$ 3 GHz frequency will not propagate through it but will get attenuated.

$$\alpha = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - \omega^2\mu\varepsilon}$$

For fundamental mode, *i.e.* TE_{10}

$$\alpha = \sqrt{\frac{\pi^2}{0.0016} - (2\pi \times 3 \times 10^9)^2 \times 4\pi \times \frac{10^{-7} \times 10^{-9}}{36\pi}}$$

$$= \pi\sqrt{625 - 4 \times 10^2}$$

$$= \pi\sqrt{225}$$

$$= 15\ \pi = 47.13\ \text{nepers/m}$$

$$= 409\ \text{db/m.}\ \textbf{Ans.}$$

Problem 4.5. *In Problem 4.4 what is minimum frequency which can be transmitted if a dielectric of $\varepsilon_r = 4$ be inserted inside the waveguide.*

Solution. $a = 4$ cm

$b = 2$ cm

$\lambda_{c(max)} = 8$ cm

$$f_{c(min)} = \frac{3 \times 10^{10}}{8}$$

$$= 3.75 \text{ GHz.}$$

Minimum frequency which can be passed without dielectric = 3.75 GHz. But if we insert a dielectric

$$\lambda_{dielectric} = \frac{\lambda_{free\ space}}{\sqrt{\varepsilon_r}}$$

$$= \frac{8}{\sqrt{4}} = 4 \text{ cm}$$

$$f_c = \frac{3 \times 10^{10}}{4}$$

$$= 7.5 \text{ GHz.}$$

Minimum frequency that can be passed with dielectric

$$= 7.5 \text{ GHz. } \textbf{Ans.}$$

Why TEM waves cannot propagate through a rectangular wave guide ?

A familiar trans electro-magnetic wave (TEM) which has the electric field, the magnetic field and the direction of propagation mutually perpendicular, cannot possibly propagate within a single conductor hollow waveguide. This is because the electric field would be short circuited by the walls since the walls are perfect conductors and hence no potential can exist across them.

Moreover, since there is no axial component of either E or H in the TEM mode, propagation through the guide is impossible. Suppose that a TEM wave exists inside a hollow guide of any shape. Then the lines of H must lie entirely in the transverse plane. Now since $\nabla \cdot H = 0$, the lines of H will be closed loops in plane perpendicular to the axis. MMF around these closed loops must be equal to the conduction or the displacement current (axial) through the loop. Now for a hollow waveguide, this axial current must be an axial displacement current which requires an axial component of E. This, however, is not present inside a TEM wave, therefore, the impossibility of propagation of TEM wave inside a hollow waveguide.